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MAM1020S **Tutorial 9** **August 2017**

1. Divide

   1. $6x^2 + x - 15$ by $3x + 5$.
      Solution: Do a long division show your work. We have:
      $$6x^2 + x - 15 = (2x - 3)(3x + 5) + 0$$
   2. $12a^4 - 17a^3 + 9a^2 + 7a - 6$ by $3a - 2$
      Solution: Do a long division show your work. We have:
      $$12a^4 - 17a^3 + 9a^2 + 7a - 6 = (4a^3 - 3a^2 + a + 3)(3a - 2) + 0$$
   3. $12a^4 - 17a^3 + 9a^2 + 7a - 4$ by $3a - 2$ (Look at your working for 2.; what will change in this example?
      Solution: Do a long division show your work. We have:
      $$12a^4 - 17a^3 + 9a^2 + 7a - 4 = (4a^3 - 3a^2 + a + 3)(3a - 2) + 2$$
      Note that $P(2/3) = 2$.
   4. $x^3 + 27$ by $x + 3$
      Solution: Do a long division show your work. We have:
      $$x^3 + 27 = (x^2 - 3x + 9)(x + 3) + 0$$
   5. $5 - 8p^3 + 12p^2$ by $1 - 2p$.
      Solution: Do a long division show your work. We have:
      $$5 - 8p^3 + 12p^2 = (4p^2 - 4p - 2)(1 - 2p) + 7$$
   6. $x^4 - y^4$ by $x - y$.
      Solution: Do a long division show your work. We have:
      $$x^4 - y^4 = (x^3 + yx^2 + y^2x + y^3)(x - y)$$
   7. $x^5 + 4x^3 - 1$ by $x^2 + 2$.
      Solution: Do a long division show your work. We have:
      $$x^5 + 4x^3 - 1 = (x^3 + 2x)(x^2 + 2) + (-1 - 4x)$$

   Give your answer as
   $$dividend = quotient \times divisor + remainder$$

2. Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$ and use the result to factor the polynomial completely.

   Solution: Do a long division show your work. We have:
   $$6x^3 - 19x^2 + 16x - 4 = (6x^2 - 7x + 2)(x - 2) = (x - 2)(2x - 1)(3x - 2)$$

3. Prove that for any $n \geq 1$,
   $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

   Solution: We argue by induction on n.

   Initialization: For $n = 1$, we have
   $$\sum_{i=1}^{n} i^3 = 1^3 = 1$$

and

$$1(1+1)/2 = 1$$

Thus

$$\sum_{i=1}^{n} i^3 = 1(1+1)/2$$

That proves that the property is true for $n = 1$.

Transmission: We suppose that the property is true for an arbitrary n that is

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 \quad (Induction\ assumption)$$

We want to prove that the property remains true for $n+1$, that is

$$\sum_{i=1}^{n+1} i^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2$$

Note that

$$\begin{array}{rcl} \sum_{i=1}^{n+1} i^3 & = & \sum_{i=1}^{n} i^3 + (n+1)^3 \\ & = & \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \text{ Using the induction assumption} \\ & = & = \frac{(n+1)^2(n^2+4(n+1))}{4} \\ & = & \frac{(n+1)^2(n+2)^2}{4} \\ & = & \left[\frac{(n+1)(n+2)}{2}\right]^2 \end{array}$$

Thus the property transmit and by the principle of induction we have proven that for any $n \geq 1$,

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

4. Evaluate

   1. $\sum_{i=1}^{n} i(4i^2 - 3)$
      Solution: We use the theorems seen in class we have

$$\begin{array}{rcl} \sum_{i=1}^{n} i(4i^2-3) & = & \sum_{i=1}^{n}(4i^3 - 3i) = 4\sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i \\ = 4\left(\frac{n(n+1)}{2}\right)^2 - 3\frac{n(n+1)}{2} & & \\ = \frac{n(n+1)(2n^2+2n-3)}{2} & & \end{array}$$

   2. $lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$.
      Solution: We use the theorems seen in class we have

$$\begin{array}{rcl} lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] & = & lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{3}{n^3} i^2 + \frac{3}{n}\right] \\ & = & lim_{n\to\infty} \left[\frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1\right] \\ & = & lim_{n\to\infty} \left[\frac{3}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n} n\right] \\ & = & lim_{n\to\infty} \left[\frac{1}{2}\cdot\frac{n}{n}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 3\right] \\ & = & lim_{n\to\infty} \left[\frac{1}{2}\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 3\right] \\ & = & \frac{1}{2}\cdot 1\cdot 2 + 3 = 4 \end{array}$$

5. Let $f(x) = sin(x)$. Find the linearization of f at $x = 0$. Use the linearization to approximate $f(0.1)$ and $f(100)$. Compare these approximations with the approximations from your calculator.

Solution: To find the linearization at 0 we need to find $f(0)$ and $f'(0)$. If $f(x) = sin(x)$, then $f(0) = sin(0) = 0$ and $f'(x) = cos(x)$, so $f'(0) = cos(0) = 1$. Thus the linearization is

$$L(x) = 0 + 1 \cdot x = x$$

We have $f(0.1) = 0.099384$, $L(0.1) = 0.1$ and $L(x) - sin(x) = 1.67 \times 10^{-4}$.

We have $f(100) = -0.506337$ and $L(100) = 100$ and $L(x) - sin(x) = 100.51$.

As expected the linearization is pretty good near 0. It is interesting to note that the error decreases very rapidly as we approach 0.

6. Let $f(x) = \sqrt{1+2x}$ and use the linearization to approximate $f(4.3)$. Find the error in the approximation of $f(4.3)$, the percentage error in the approximation of $f(4.3)$ and the percentage error in the approximation of Δf.

   Solution: Note that it is easy to compute $f(4) = \sqrt{9} = 3$. We compute the linearization of f at 4. To do this we need $f(4) = 9$ and we need to find $f'(4)$. We compute the derivative by the power rule and chain rule,

   $$f'(x) = 1/2(1+2x)^{-1/2} \times 2 = \frac{1}{\sqrt{1+2x}}$$

   Evaluating at $x = 4$, give $f'(4) = 1/3$. Thus the linearization of f at 4 is

   $$L(x) = f(4) + f'(4)(x-4) = 3 + 1/3(x-4)$$

   To make the approximation we write

   $$f(4.3) \sim L(4.3) = f(4) + 1/30.3 = 3.1$$

   A calculator gives the more accurate approximation

   $$f(4.3) = 3.0984$$

   The errors are

   $$|f(4.3) - 3.1| \sim 0.00161$$
   $$100\% \times \frac{|f(4.3) - 3.1|}{|f(4.3)|} \sim 0.0521\%$$

7. Suppose that a curve is given by the equation $x^2 + y^3 = 2x^2y$. Verify that the point $(x, y) = (1, 1)$ lies on the curve. Assume that the curve is given by a function $y = y(x)$ for x near 1 and approximate $y(1.2)$.

   Solution : To verify that $(x, y) = (1, 1)$ lies on the curve, we need to know that $1^3 + 1^2 = 2 \cdot 1^2$ which is true.

   To find the linearization, we use that $y(1) = 1$ and find the derivative of y at $x = 1$. Differentiating

   $$\frac{d}{dx}(x^2 + y^3) = fracddx(2x^2y)$$

   gives

   $$2x + 3y^2y' = 4y + 2x^2y'$$

   Solving for y' gives

   $$y' = \frac{4y - 2x}{3y^2 - 2x^2}$$

   and that $y'(1) = 2$. Thus the linearization of y is $L(x) = 1+2(x?1)$ and $L(1.2) \sim 1.4$. Thus the point $(1, 1.2)$ should be close to the curve. If we substitute this point into the equation $x^2 + y^3 = 2x^2y$, we find that $1.2^2 + 1.4^3 = 4.184$ and $21.2^2 1.4 = 4.0320$. As these values are close, we expect that $(1.2, 1.4)$ is close to a point on the curve.

Review part:

1. Consider the following functions

   $$f(x) = \frac{cos(x)}{(1+x^2)^2},\ g(x) = \sqrt{3 - e^x}\ and\ h(x) = \frac{1}{x}$$

(a) What is the range of $f(x)$?

(b) Is f even or odd of neither.

(c) Show that g, g^{-1}. What is the domain of g^{-1}?

(d) Find $h \circ g$ and state its domains.

(e) Starting with $h(x) = \frac{1}{x}$, write down the new algebraic function k obtained after stretching it vertically by 2 units then shifting it down by 3 units and to the right by 4 units. Now quickly sketch the new function, k.

2. Find the parametric equations for the path of the particle that moves counter-clockwise halfway around the ellipse $9(x-2)^2 + 4y^2 = 36$ from left to right. Determine the foci on the Cartesian plane.

3. "The cost of living continues to rise, but at a slower rate." What does this statement mean, in terms of a function and its first and second derivatives?

4. Consider the function $s(t) = t^2 - t - 2$, where $s(t)$ is displacement (in meters) and t is time (in seconds).

   (a) Determine the average speed over the first 4 seconds.

   (b) Use the definition of the derivative at $t = 2$ to determine the instantaneous speed at that moment.

   (c) Given $g(t) = \frac{s(t)}{t-2}$ and determine $lim_{t\to 2} g(t)$.

   (d) Is $g(t)$ continuous at $t = 2$? Is $g(t)$ differentiable at $t = 2$? Give reason for your answers.

5. Find $\frac{dy}{dx}$ for the following

   (a) $y = (tan(x))^3$

   (b) $y = (cot(x))^x$

   (c) $3^{\frac{x}{y}} = e^{-x^2}$

6. (a) Solve the equation $log_{10}(\sqrt{x^2+3x+2}) = 1$;

   (b) Write function $f(x) = |x+1| + |x-3|$ as a piecewise defined function and then sketch its graph.

7. State whether the following statements are True or False. Justify your answers by providing a proof if your answers by providing a proof if your answer is True and a counterexample if it is false.

   (a) $lim_{x\to 1}(x^3-1)^2 cos\left(\frac{1}{x-1}\right)$

   (b) $lim_{x\to 0^+} \frac{ln(cos(x))}{x^2}$

8. Find the derivative $y' = \frac{dy}{dx}$ of the following functions

   (a) $y = f(x) = x^e + x^{cos(x)}$

   (b) $y = g(y) = ln\left(\sqrt{\frac{sin(x)}{x^2+1}}\right)$

9. Let C be the curve given by the parametric equations

$$x = t^2,\ y = (2-t)^2,\ t \in \mathbb{R}$$

   (a) Find the derivative $\frac{dy}{dx}$.

   (b) Find all points on C where the tangent line is horizontal.

   (c) Find all points on C where the tangent line is vertical.

   (d) Find the equation of the line tangent to the curve C at the point corresponding to $t = 4$.

10. In this question, do not simplify your answer.

   (a) Write down $f'(x)$ for $f(x) = e^{xtan(x)}$.

(b) Write down $g'(x)$ for $g(x) = sin^4\left(\frac{ln(x)}{\sqrt{x}}\right)$.

11. Assume that the following equation defines y as function of x

$$x^3 - 2xy^2 + y^3 = 1 \ (*)$$

(a) Find $\frac{dy}{dx}$.

(b) Find the equation of the tangent line to the curve defined by $(*)$ at the point $(1, 2)$.

12. Suppose

$$f(x) = \begin{cases} x^2 + 11x + 10 \ (x \leq 1) \\ a + bx - x^2 \ (x > 1) \end{cases}$$

If a, b are such that f is differentiable at $x = 1$ then $a + b$ is equal to ?

13. If a product is priced at Rx then $D(x)$ units will sell, where $D(x) = 4800 - 3x\sqrt{x}$ and $22 \leq x \leq 91$. Find $D'(30)$. Also find the actual change in sales if the price is increased from $R30$ to $R31$. What is the sum of the numerical values of the two answer (to 2 decimals, not rounded)?

14. A factory emits nitrous oxide. The concentration in the surrounding air at a distance of $x\ km$ from the factory is estimated as $C(x) = \frac{2}{x^2}$ parts per millions for $x \geq 1$. Find

(a) The instantaneous rate of change of concentration with respect to distance at a point A which is 10 km from the factory.

(b) The exact change in concentration if one measures first at A and then at a point B which is 1 km further away.

What is the sum of the two answers?

15. For all but two choices of a, the function

$$f(x) = \begin{cases} (x - 18)^3 + 2 \ x \leq a \\ (x - 18)^2 + 2 \ x > a \end{cases}$$

will be discontinuous at the point $x = a$. Find the sum of the value(s) of a such that $f(x)$ is continuous at $x = a$.

16. Find the linear approximation of the function $f(x) = \sqrt{3x + 7}$ at $x = 3$ and use it to approximate $f(5)$.

17. Suppose $f(x) = \frac{1}{x+2}$ and that the point P has coordinates $(4, -1)$. Find the two points on the graph of f which are such that the lines joining these points to P are tangent to the graph. Which of the following is the sum of x-coordinates of these two points?

18. The function $f(x) = 3x^4 + 6x^3 + 3x^2 + 2x + 1$ is concave down on the interval (p, q). Which of the following is the right endpoint q of this interval?

19. Suppose $y = \frac{3x}{7x^2+4}$. Which of the following is the y-intercept of the line which is normal to the curve at the point where $x = 2$? (normal means perpendicular to the tangent.)

20. Find A, B, C so that the curve $y = Ax^2 + Bx + C$ will pass through $(1, 2)$ and be tangent to the line $2x + y = 7$ at $(2, 3)$. Which of the following is $A + B$?

21. Suppose that $f(x) = \frac{2x+3}{5x+1}$ and that $g(x) = (\sqrt{x} + 1)(x^4 + 3x + 2)(x + 3)$. What is the value $f'(1) + g'(1)$?

22. If $f(x) = \frac{3x+2}{2x^2+1}$ then $f'(1) = \cdots$.

23. If $f(x) = (\sqrt{x} + 4x + 2x^2)(x^3 + 5x + 5)$ then $f'(1)$ is equal to $\cdots$

24. If $f(x) = (3\sqrt{x} - 4)^4(3x^2 + 1)$, then $f'(1)$ is equal to $\cdots$

25. If $y = \frac{x^4}{5-x}$, find $y''(1)$.

26. If $f(x) = (4x + 3)^{2/3}$, find $f'(1)$.

27. If $f(x) = (x^{3/5} + 2)^4$, find $f'(1)$.

28. If $f(x) = (5x^3 + 3)^{2/3}x^3$, find $f'(1)$.

29. If $f(x) = \frac{e^{\sqrt{5x^3+3}}}{2x+3}$, find $f'(1)$.

30. One of the turning points of the graph of

$$y = \frac{2x^2}{(x+3)(x-2)}$$

is at $(0, 0)$. Find the y-coordinate of the other turning point.